CHAPTER 5

DERIVATIVES

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Compute the derivative of a constant.
2. Compute the derivative of a variable raised to a power.
3. Compute the derivative of the sum and product of two or more functions and the quotient of two functions.
4. Compute the derivative of a function raised to a power, in radical form, and by using the chain rule.
5. Compute the derivative of an inverse function, an implicit function, a trigonometric function, and a natural logarithmic function.
6. Compute the derivative of a constant raised to a variable power.

INTRODUCTION

In the previous chapter on limits, we used the delta process to find the limit of a function as Δx approached zero. We called the result of this tedious and, in some cases, lengthy process the derivative. In this chapter we will examine some rules used to find the derivative of a function without using the delta process.

To find how y changes as x changes, we take the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \rightarrow 0$ and write

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

which is called the derivative of y with respect to x; we use the symbol $\frac{dy}{dx}$ to indicate the derivative and write

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

In this section we will learn a number of rules that will enable us to easily obtain the derivative of many algebraic functions. In the derivation of these rules, which will be called theorems, we will assume that

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

exists and is finite.

DERIVATIVE OF A CONSTANT

The method we will use to find the derivative of a constant is similar to the delta process used in the previous chapter but includes an analytical proof. A diagram is used to give a geometrical meaning of the function.

Theorem 1. *The derivative of a constant is zero*. Expressed as a formula, this may be written as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

where $y = c$.

PROOF: In figure 5-1, the graph of

$$y = c$$

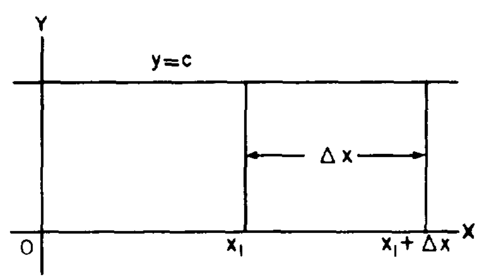

Figure 5-1.—Graph of y = c, where c is a constant.

where c is a constant, the value of y is the same for all values of x, and any change in x (that is, Δx) does not affect y; then

$$\Delta y = c - c = 0$$

$$\frac{\Delta y}{\Delta x} = 0$$

and

$$\frac{dy}{dx} = 0$$

Another way of stating this is that when x is equal to x_1 and when x is equal to $x_1 + \Delta x$, y has the same value. Therefore,

$$y = c$$

and

$$y + \Delta y = c$$

so that

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{c - c}{\Delta x}$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

The equation

$$y = c$$

represents a straight line parallel to the X axis. The slope of this line will be zero for all values of x. Therefore, the derivative is zero for all values of x.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$y = 6$$

SOLUTION:

$$y = 6$$

and

$$y + \Delta y = 6$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{6 - 6}{\Delta x} \\ &= 0\end{aligned}$$

DERIVATIVES OF VARIABLES

In this section of variables, we will extend the theorems of limits covered previously. Recall that a derivative is actually a limit. The proof of the theorems presented here involve the delta process.

POWER FORM

Theorem 2. *The derivative of the function*

$$y = x^n$$

is given by

$$\frac{dy}{dx} = nx^{n-1}$$

if n is any real number.

PROOF: By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - (x)^n}{\Delta x}$$

The expression $(x + \Delta x)^n$ may be expanded by the binomial theorem into

$$x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \cdots + \Delta x^n$$

Substituting in the expression for the derivative, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \cdots + \Delta x^n}{\Delta x}$$

Simplifying, this becomes

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + \Delta x^{n-1}\right]$$

Letting Δx approach zero, we have

$$\frac{dy}{dx} = nx^{n-1}$$

Thus, the proof is complete.

EXAMPLE: Find the derivative of

$$f(x) = x^5$$

SOLUTION: Apply Theorem 2, such that,

$$x^5 = x^n$$

Therefore,

$$n = 5$$

and

$$n - 1 = 4$$

so that given

$$\frac{dy}{dx} = nx^{n-1}$$

and substituting values for n, find that

$$\frac{dy}{dx} = 5x^4$$

EXAMPLE: Find the derivative of

$$f(x) = x$$

SOLUTION: Apply Theorem 2, such that,

$$x^n = x$$

Therefore,

$$n = 1$$

and

$$n - 1 = 0$$

so that

$$\frac{dy}{dx} = x^0$$

$$= 1$$

The previous example is a special case of the power form and indicates that the derivative of a function with respect to itself is 1.

EXAMPLE: Find the derivative of

$$f(x) = ax$$

where a is a constant.

SOLUTION:

$$f(x) = ax$$

and

$$f(x + \Delta x) = a(x + \Delta x)$$

$$= ax + a\Delta x$$

so that

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= (ax + a\Delta x) - ax$$

$$= a\Delta x$$

Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x}$$

$$= a$$

Table 5-1.—Derivatives of Functions

$f(x)$	3	x	x^2	x^3	x^4	$3x^2$	$9x^3$	x^{-1}	x^{-2}	$3x^{-4}$
$\frac{dy}{dx}$	0	1	$2x$	$3x^2$	$4x^3$	$6x$	$27x^2$	$-x^{-2}$	$-2x^{-3}$	$-12x^{-5}$

The previous example is a continuation of the derivative of a function with respect to itself and indicates that the derivative of a function with respect to itself, times a constant, is that constant.

EXAMPLE: Find the derivative of

$$f(x) = 6x$$

SOLUTION:

$$\frac{dy}{dx} = 6$$

A study of the functions and their derivatives in table 5-1 should further the understanding of this section.

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = 21$
2. $f(x) = x$
3. $f(x) = 21x$
4. $f(x) = 7x^3$
5. $f(x) = 4x^2$
6. $f(x) = 3x^{-2}$

ANSWERS:

1. 0
2. 1
3. 21
4. $21x^2$
5. $8x$
6. $-6x^{-3}$

SUMS

Theorem 3. *The derivative of the sum of two or more differentiable functions of x is equal to the sum of their derivatives.*

If two functions of x are given, such that

$$u = g(x)$$

and

$$v = h(x)$$

and also

$$y = u + v$$

$$= g(x) + h(x)$$

then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

PROOF:

$$y = g(x) + h(x) \tag{5.1}$$

and

$$y + \Delta y = g(x + \Delta x) + h(x + \Delta x) \tag{5.2}$$

Subtract equation (5.1) from equation (5.2):

$$\Delta y = g(x + \Delta x) + h(x + \Delta x) - g(x) - h(x)$$

Rearrange this equation such that

$$\Delta y = g(x + \Delta x) - g(x) + h(x + \Delta x) - h(x)$$

Divide both sides of the equation by Δx and then take the limit as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$+ \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$$

But, by definition

$$\lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = \frac{du}{dx}$$

and

$$\lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} = \frac{dv}{dx}$$

Then by substitution,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = x^3 - 8x^2 + 7x - 5$$

SOLUTION: Theorem 3 indicates that we should find the derivative of each term and then show them as a sum; that is, if

$$y = x^3, \frac{dy}{dx} = 3x^2$$

$$y = -8x^2, \frac{dy}{dx} = -16x$$

$$y = 7x, \frac{dy}{dx} = 7$$

$$y = -5, \frac{dy}{dx} = 0$$

and

$$y = x^3 - 8x^2 + 7x - 5$$

then

$$\frac{dy}{dx} = 3x^2 - 16x + 7 + 0$$

$$= 3x^2 - 16x + 7$$

PRACTICE PROBLEMS:

Find the derivative of the following:

1. $f(x) = x^2 + x - 1$
2. $f(x) = 2x^4 + 3x + 16$
3. $f(x) = 2x^3 + 3x^2 + x - 3$
4. $f(x) = 3x^3 + 2x^2 - 4x + 2 + 2x^{-1} - 3x^{-3}$

ANSWERS:

1. $2x + 1$
2. $8x^3 + 3$
3. $6x^2 + 6x + 1$
4. $9x^2 + 4x - 4 - 2x^{-2} + 9x^{-4}$

PRODUCTS

Theorem 4. *The derivative of the product of two differentiable functions of x is equal to the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function.*

If

$$y = uv$$

then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This theorem may be extended to include the product of three differentiable functions or more. The result for three functions would be as follows:

If

$$y = uvw$$

then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$

EXAMPLE: Find the derivative of

$$f(x) = (x^2 - 2)\ (x^4 + 5)$$

SOLUTION: The derivative of the first factor is $2x$, and the derivative of the second factor is $4x^3$. Therefore,

$$f'(x) = (x^2 - 2)\ (4x^3) + (x^4 + 5)\ (2x)$$

$$= 4x^5 - 8x^3 + 2x^5 + 10x$$

$$= 6x^5 - 8x^3 + 10x$$

EXAMPLE: Find the derivative of

$$f(x) = (x^3 - 3)\ (x^2 + 2)\ (x^4 - 5)$$

SOLUTION: The derivatives of the three factors, in the order given, are $3x^2$, $2x$, and $4x^3$.

Therefore,

$$f'(x) = (x^3 - 3)\ (x^2 + 2)\ (4x^3)$$

$$+ (x^2 + 2)\ (x^4 - 5)\ (3x^2)$$

$$+ (x^3 - 3)\ (x^4 - 5)\ (2x)$$

Expanding, we get

$$f'(x) = 4x^8 + 8x^6 - 12x^5 - 24x^3$$
$$+ 3x^8 + 6x^6 - 15x^4 - 30x^2$$
$$+ 2x^8 - 6x^5 - 10x^4 + 30x$$
$$= 9x^8 + 14x^6 - 18x^5 - 25x^4 - 24x^3 - 30x^2 + 30x$$

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = x^3(x^2 - 4)$
2. $f(x) = (x^3 - 3)(x^2 + 2x)$
3. $f(x) = (x^2 - 7x)(x^5 - 4x^2)$
4. $f(x) = (x - 2)(x^2 - 3)(x^3 - 4)$

ANSWERS:

1. $5x^4 - 12x^2$
2. $5x^4 + 8x^3 - 6x - 6$
3. $7x^6 - 42x^5 - 16x^3 + 84x^2$
4. $6x^5 - 10x^4 - 12x^3 + 6x^2 + 16x + 12$

QUOTIENTS

Theorem 5. *At a point where the denominator is not equal to zero, the derivative of the quotient of two differentiable functions of x is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

If

$$y = \frac{u}{v}$$

then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

EXAMPLE: Find the derivative of the function

$$f(x) = \frac{x^2 - 7}{2x + 8}$$

SOLUTION: The derivative of the numerator is $2x$, and the derivative of the denominator is 2. Therefore,

$$f'(x) = \frac{(2x + 8)\,(2x) - (x^2 - 7)\,(2)}{(2x + 8)^2}$$

$$= \frac{4x^2 + 16x - 2x^2 + 14}{(2x + 8)^2}$$

$$= \frac{2x^2 + 16x + 14}{4(x + 4)^2}$$

$$= \frac{x^2 + 8x + 7}{2(x + 4)^2}$$

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = \dfrac{x^4}{x^2 - 2}$

2. $f(x) = \dfrac{x^2 - 3}{x + 7}$

3. $f(x) = \dfrac{x^2 + 3x + 5}{x^3 - 4}$

ANSWERS:

1. $\dfrac{2x^5 - 8x^3}{(x^2 - 2)^2}$

2. $\dfrac{x^2 + 14x + 3}{(x + 7)^2}$

3. $\dfrac{-(x^4 + 6x^3 + 15x^2 + 8x + 12)}{(x^3 - 4)^2}$

POWERS OF FUNCTIONS

Theorem 6. *The derivative of any differentiable function of x raised to the power n, where n is any real number, is equal to n times the polynomial function of x to the (n − 1) power times the derivative of the polynomial itself.*

If

$$y = u^n$$

where u is any differentiable function of x, then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = (x^3 - 3x^2 + 2x)^7$$

SOLUTION: Apply Theorem 6 and find

$$\frac{dy}{dx} = 7(x^3 - 3x^2 + 2x)^6\,(3x^2 - 6x + 2)$$

EXAMPLE: Find the derivative of the function

$$f(x) = \frac{(x^2 + 2)^3}{x - 1}$$

SOLUTION: This problem involves Theorem 5 and Theorem 6. Theorem 6 is used to find the derivative of the numerator; then Theorem 5 is used to find the derivative of the resulting quotient.

The derivative of the numerator is

$$3(x^2 + 2)^2 (2x)$$

and the derivative of the denominator is 1. Then, by Theorem 5

$$\frac{dy}{dx} = \frac{(x - 1)[3(x^2 + 2)^2 (2x)] - (1) (x^2 + 2)^3}{(x - 1)^2}$$

$$= \frac{6x(x^2 + 2)^2 (x - 1) - (x^2 + 2)^3}{(x - 1)^2}$$

$$= \frac{(x^2 + 2)^2 [6x(x - 1) - (x^2 + 2)]}{(x - 1)^2}$$

$$= \frac{(x^2 + 2)^2 (5x^2 - 6x - 2)}{(x - 1)^2}$$

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = (x^3 + 2x - 6)^2$
2. $f(x) = 5(x^2 + x + 7)^4$
3. $f(x) = \dfrac{2(x + 3)^3}{3x}$

ANSWERS:

1. $2(x^3 + 2x - 6) (3x^2 + 2)$
2. $20(x^2 + x + 7)^3 (2x + 1)$
3. $\dfrac{18x(x + 3)^2 - 6(x + 3)^3}{9x^2}$

RADICALS

To differentiate a function containing a radical, replace the radical by a fractional exponent; then find the derivative by applying the appropriate theorems.

EXAMPLE: Find the derivative of

$$f(x) = \sqrt{2x^2 - 5}$$

SOLUTION: Replace the radical by the proper fractional exponent, such that

$$f(x) = (2x^2 - 5)^{1/2}$$

and by Theorem 6

$$\frac{dy}{dx} = \frac{1}{2}(2x^2 - 5)^{(1/2)-1}(4x)$$

$$= \frac{1}{2}(2x^2 - 5)^{-1/2}(4x)$$

$$= 2x(2x^2 - 5)^{-1/2}$$

$$= \frac{2x}{\sqrt{2x^2 - 5}}$$

$$= \frac{2x\sqrt{2x^2 - 5}}{2x^2 - 5}$$

EXAMPLE: Find the derivative of

$$f(x) = \frac{2x + 1}{\sqrt{3x^2 + 2}}$$

SOLUTION: Replace the radical by the proper fractional exponent, thus

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}}$$

At this point a decision is in order. This problem may be solved by either writing

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}} \qquad (1)$$

and applying Theorem 6 in the denominator and then applying Theorem 5 for the quotient or writing

$$f(x) = (2x + 1)\,(3x^2 + 2)^{-1/2} \tag{2}$$

and applying Theorem 6 for the second factor and then applying Theorem 4 for the product.

The two methods of solution are completed individually as follows:

Use equation (1):

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}}$$

Find the derivative of the denominator

$$\frac{d}{dx}(3x^2 + 2)^{1/2}$$

by applying the power theorem

$$\frac{d}{dx}(3x^2 + 2)^{1/2} = \frac{1}{2}(3x^2 + 2)^{(1/2)-1}(6x)$$

$$= 3x(3x^2 + 2)^{-1/2}$$

The derivative of the numerator is

$$\frac{d}{dx}(2x + 1) = 2$$

Now apply Theorem 5:

$$f'(x) = \frac{(3x^2 + 2)^{1/2}\,(2) - (2x + 1)\,[3x(3x^2 + 2)^{-1/2}]}{(3x^2 + 2)}$$

Multiply both numerator and denominator by

$$(3x^2 + 2)^{1/2}$$

and simplify:

$$f'(x) = \frac{2(3x^2 + 2) - 3x(2x + 1)}{(3x^2 + 2)^{3/2}}$$

$$= \frac{6x^2 + 4 - 6x^2 - 3x}{(3x^2 + 2)^{3/2}}$$

$$= \frac{4 - 3x}{(3x^2 + 2)^{3/2}}$$

To find the same solution by a different method, use equation (2):

$$f(x) = (2x + 1)\ (3x^2 + 2)^{-1/2}$$

Find the derivative of each factor:

$$\frac{d}{dx}(2x + 1) = 2$$

and

$$\frac{d}{dx}(3x^2 + 2)^{-1/2} = -1/2(3x^2 + 2)^{(-1/2)-1}(6x)$$

$$= -3x(3x^2 + 2)^{-3/2}$$

Now apply Theorem 4:

$$f'(x) = (2x + 1)\ \left[-3x(3x^2 + 2)^{-3/2}\right] + (3x^2 + 2)^{-1/2}(2)$$

Multiply both numerator and denominator by

$$(3x^2 + 2)^{3/2}$$

such that,

$$f'(x) = \frac{-3x(2x + 1) + 2(3x^2 + 2)}{(3x^2 + 2)^{3/2}}$$

$$= \frac{-6x^2 - 3x + 6x^2 + 4}{(3x^2 + 2)^{3/2}}$$

$$= \frac{4 - 3x}{(3x^2 + 2)^{3/2}}$$

which agrees with the solution of the first method used.

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = \sqrt{x}$

2. $f(x) = \dfrac{1}{\sqrt{x}}$

3. $f(x) = \sqrt{3x - 4}$

4. $f(x) = \sqrt[3]{4x^2 - 3x + 2}$

ANSWERS:

1. $\dfrac{1}{2\sqrt{x}}$ or $\dfrac{\sqrt{x}}{2x}$

2. $-\dfrac{1}{2\sqrt{x^3}}$ or $-\dfrac{\sqrt{x^3}}{2x^3}$

3. $\dfrac{3}{2\sqrt{3x - 4}}$ or $\dfrac{3\sqrt{3x - 4}}{2(3x - 4)}$

4. $\dfrac{8x - 3}{3\sqrt[3]{(4x^2 - 3x + 2)^2}}$ or $\dfrac{(8x - 3)\sqrt[3]{4x^2 - 3x + 2}}{3(4x^2 - 3x + 2)}$

CHAIN RULE

A frequently used rule in differential calculus is the chain rule. This rule links together derivatives that have related variables. The *chain rule* is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where the variable $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x.

EXAMPLE: Find the derivative of

$$y = (x + x^2)^2$$

SOLUTION: Let

$$u = (x + x^2)$$

and

$$y = u^2$$

Then,

$$\frac{dy}{du} = 2u$$

and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\frac{du}{dx} \tag{1}$$

Now,

$$\frac{du}{dx} = 1 + 2x$$

and substituting into equation (1) gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u(1 + 2x)$$

but,

$$u = (x + x^2)$$

Therefore,

$$\frac{dy}{dx} = 2(x + x^2)\ (1 + 2x)$$

EXAMPLE: Find $\frac{dy}{dx}$ where

$$y = 12t^4 + 7t$$

and

$$t = x^2 + 4$$

SOLUTION: By the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$\frac{dy}{dt} = 48t^3 + 7$$

and

$$\frac{dt}{dx} = 2x$$

Then,

$$\frac{dy}{dx} = (48t^3 + 7)\ (2x)$$

and by substitution

$$\frac{dy}{dx} = [48(x^2 + 4)^3 + 7]\ (2x)$$

PRACTICE PROBLEMS:

Find $\frac{dy}{dx}$ in the following:

1. $y = 3t^3 + 8t$ and

   $t = x^3 + 2$

2. $y = 7n^2 + 8n + 3$ and

   $n = 2x^3 + 4x^2 + x$

ANSWERS:

1. $[9(x^3 + 2)^2 + 8]\ (3x^2)$

2. $[14(2x^3 + 4x^2 + x) + 8]\ (6x^2 + 8x + 1)$

INVERSE FUNCTIONS

Theorem 7. *The derivative of an inverse function is equal to the reciprocal of the derivative of the direct function.*

In the equations to this point, x has been the independent variable and y has been the dependent variable. The equations have been in a form such as

$$y = x^2 + 3x + 2$$

Suppose that we have a function like

$$x = \frac{1}{y^2} - \frac{1}{y}$$

and we wish to find the derivative $\frac{dy}{dx}$. Notice that if we solve for y in terms of x, using the quadratic formula, we get the more complicated function:

$$y = \frac{-1 \pm \sqrt{1 + 4x}}{2x}$$

If we call this function the direct function, then

$$x = \frac{1}{y^2} - \frac{1}{y}$$

is the inverse function. To determine $\frac{dy}{dx}$ from the inverse function is easy.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$x = \frac{1}{y^2} - \frac{1}{y}$$

SOLUTION: The derivative $\frac{dx}{dy}$ is

$$\frac{dx}{dy} = -2y^{-3} + y^{-2}$$

$$= \frac{-2}{y^3} + \frac{1}{y^2}$$

$$= \frac{-2 + y}{y^3}$$

The reciprocal of $\frac{dx}{dy}$ is the derivative $\frac{dy}{dx}$ of the direct function, and we find

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{y^3}{y - 2}$$

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$x = y^2$$

SOLUTION: Find $\frac{dx}{dy}$ to be

$$\frac{dx}{dy} = 2y$$

Then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y}$$

PRACTICE PROBLEMS:

Find the derivative $\frac{dy}{dx}$ of the following functions:

1. $x = 4 - y^2$
2. $x = y^2 + 9y$

ANSWERS:

1. $-\frac{1}{2y}$
2. $\frac{1}{2y + 9}$

IMPLICIT FUNCTIONS

In equations containing x and y, separating the variables is not always easy. If we do not solve an equation for y, we call y an *implicit function* of x. In the equation

$$x^2 - 4y = 0$$

y is an implicit function of x, and x is also called an implicit function of y. If we solve this equation for y, that is

$$y = \frac{x^2}{4}$$

then y would be called an explicit function of x. In many cases such a solution would be far too complicated to handle conveniently.

When y is given by an equation such as

$$y^2 + xy^2 = 2$$

y is an implicit function of x.

Whenever we have an equation of this type in which y is an implicit function of x, we can differentiate the function in a straightforward manner. The derivative of each term containing y will be followed by $\frac{dy}{dx}$. Refer to Theorem 6.

EXAMPLE: Obtain the derivative $\frac{dy}{dx}$ of

$$y^2 + xy^2 = 2$$

SOLUTION: Find the derivative of y^2:

$$\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}$$

the derivative of xy^2:

$$\frac{d(xy^2)}{dx} = x(2y)\frac{dy}{dx} + (y^2)(1)$$

and the derivative of 2:

$$\frac{d(2)}{dx} = 0$$

such that,

$$2y\frac{dy}{dx} + 2xy\frac{dy}{dx} + y^2 = 0$$

Solving for $\frac{dy}{dx}$ we find that

$$(2y + 2xy)\frac{dy}{dx} = -y^2$$

and

$$\frac{dy}{dx} = \frac{-y^2}{2y + 2xy}$$

$$= \frac{-y}{2 + 2x}$$

Thus, whenever we differentiate an implicit function, the derivative will usually contain terms in both x and y.

PRACTICE PROBLEMS:

Find the derivative $\frac{dy}{dx}$ of the following:

1. $x^5 + 4xy^3 - 3y^5 = 0$

2. $x^3y^2 = 3xy$

3. $x^2y + y^3 = 4$

ANSWERS:

1. $\frac{-5x^4 - 4y^3}{12xy^2 - 15y^4}$

2. $\frac{3x^2y^2 - 3y}{3x - 2x^3y}$

3. $\frac{-2xy}{x^2 + 3y^2}$

TRIGONOMETRIC FUNCTIONS

If we are given

$$y = \sin u$$

we may state that, from the general formula,

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\sin (u + \Delta u) - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin u \cos \Delta u + \cos u \sin \Delta u - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin u(\cos \Delta u - 1)}{\Delta u} + \lim_{\Delta u \to 0} \frac{\sin \Delta u \cos u}{\Delta u} \qquad (5.3)$$

Since

$$\lim_{\Delta u \to 0} \frac{\cos \Delta u - 1}{\Delta u} = 0 \qquad (5.4)$$

and

$$\lim_{\Delta u \to 0} \frac{\sin \Delta u}{\Delta u} = 1 \qquad (5.5)$$

Then by substituting equations (5.4) and (5.5) into equation (5.3),

$$\frac{dy}{du} = \cos u \qquad (5.6)$$

Now we are interested in finding the derivative $\frac{dy}{dx}$ of the function sin u, so we apply the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

From the chain rule and equation (5.6), we find

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In other words, to find the derivative of the sine of a function, we use the cosine of the function times the derivative of the function.

By a similar process we find the derivative of the cosine function to be

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$$

The derivatives of the other trigonometric functions may be found by expressing them in terms of the sine and cosine. That is,

$$\frac{d(\tan u)}{dx} = \frac{d}{dx}\left(\frac{\sin u}{\cos u}\right)$$

and by substituting sin u for u, cos u for v, and du for dx in the expression of the quotient theorem

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

we have

$$\frac{dy}{du} = \frac{d}{du}\left(\frac{\sin u}{\cos u}\right)$$

$$= \frac{\cos u \frac{d(\sin u)}{du} - \sin u \frac{d(\cos u)}{du}}{\cos^2 u} \qquad (5.7)$$

Taking

$$\frac{d(\sin u)}{du} = \cos u$$

and

$$\frac{d(\cos u)}{du} = -\sin u$$

and substituting into equation (5.7), we find that

$$\frac{dy}{du} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u}$$

$$= \frac{1}{\cos^2 u}$$

$$= \sec^2 u \qquad (5.8)$$

Now using the chain rule and equation (5.8), we find

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \sec^2 u \frac{du}{dx}$$

By stating the other trigonometric functions in terms of the sine and cosine and using similar processes, we may find the following derivatives:

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = \sin 3x$$

SOLUTION:

$$\frac{dy}{dx} = \cos 3x \frac{d(3x)}{dx}$$

$$= 3 \cos 3x$$

EXAMPLE: Find the derivative of the function

$$y = \tan^2(3x)$$

SOLUTION: Use the power theorem to find

$$\frac{dy}{dx} = 2 \tan 3x \frac{d(\tan 3x)}{dx}$$

Then find

$$\frac{d}{dx}(\tan 3x) = \sec^2 3x \frac{d}{dx}(3x)$$

and

$$\frac{d}{dx}(3x) = 3$$

Combining all of these, we find that

$$\frac{dy}{dx} = (2 \tan 3x)(\sec^2 3x)(3)$$

$$= 6 \tan 3x \sec^2 3x$$

PRACTICE PROBLEMS:

Find the derivative of the following:

1. $y = \sin 2x$
2. $y = (\cos x^2)^2$

ANSWERS:

1. $2 \cos 2x$
2. $-4x \cos x^2 \sin x^2$

NATURAL LOGARITHMIC FUNCTIONS

Theorem 8. *The natural logarithm*

$$y = \ln x$$

has the derivative

$$\frac{dy}{dx} = \frac{1}{x} \tag{5.9}$$

for $x > 0$.

If u is a positive differentiable function of x, then by (5.9) and the chain rule,

$$\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = \ln (6 - x^2)$$

SOLUTION:

$$\frac{dy}{dx} = \left(\frac{1}{6 - x^2}\right)\frac{d(6 - x^2)}{dx}$$

$$= \left(\frac{1}{6 - x^2}\right)(-2x)$$

$$= \frac{-2x}{6 - x^2} \text{ or } \frac{2x}{x^2 - 6}$$

DERIVATIVE OF CONSTANTS TO VARIABLE POWERS

In this section two forms of a constant to a variable power will be presented. The two exponential functions will be e^x and a^x, where x is the variable, a is any constant, and e is equal to 2.71828. . . .

Recalling our study of logarithms in *Mathematics,* Volume 2-A, since ln and e are inverse functions, then

$$\ln (e) = 1$$

$$\ln (e^x) = x \qquad (5.10)$$

and

$$\ln (a^x) = x \ln a$$

If

$$y = e^x$$

then

$$\frac{dy}{dx} = e^x \qquad (5.11)$$

PROOF: Since $y = \ln x$ is differentiable, so is its inverse, $y = e^x$. To obtain the derivative of $y = e^x$, we differentiate both sides of equation (5.10) with respect to x, which gives

$$\frac{1}{e^x}\frac{d(e^x)}{dx} = 1 \qquad (5.12)$$

Multiplying both sides of equation (5.12) by e^x gives

$$\frac{d(e^x)}{dx} = e^x$$

Chain rule differentiation and equation (5.11) give

$$\frac{d(e^u)}{dx} = e^u\frac{du}{dx}$$

EXAMPLE: Find $\frac{dy}{dx}$ for $y = e^{6x}$.

SOLUTION:

$$\frac{dy}{dx} = e^{6x}\frac{d(6x)}{dx}$$

$$= e^{6x}(6) \text{ or } 6e^{6x}$$

EXAMPLE: Find $\frac{dy}{dx}$ for $y = e^{\sin x}$.

SOLUTION:

$$\frac{dy}{dx} = e^{\sin x}\frac{d(\sin x)}{dx}$$

$$= e^{\sin x}\cos x$$

If

$$y = a^x$$

then

$$\frac{dy}{dx} = (\ln a)a^x \qquad (5.13)$$

PROOF: Applying logarithmic rules,

$$a^x = e^{\ln (a^x)}$$

$$= e^{x \ln a} \tag{5.14}$$

Differentiating both sides of equation (5.14) gives

$$\frac{d(a^x)}{dx} = \frac{d(e^{x \ln a})}{dx}$$

$$= e^{x \ln a}\frac{d(x \ln a)}{dx}$$

$$= e^{x \ln a}(\ln a)$$

$$= e^{\ln (a^x)}(\ln a)$$

$$= a^x(\ln a) \text{ or } (\ln a)a^x$$

NOTE: $\ln a$ is a constant.

EXAMPLE: Find $\frac{dy}{dx}$ of $y = 2^x$.

SOLUTION:

$$\frac{dy}{dx} = (\ln 2)2^x$$

Chain rule differentiation and equation (5.13) give

$$\frac{d(a^u)}{dx} = (\ln a)a^u \frac{du}{dx}$$

EXAMPLE: Find $\frac{dy}{dx}$ of $y = 3^{(x^2 - 1)}$.

SOLUTION:

$$\frac{dy}{dx} = (\ln 3)3^{(x^2 - 1)}\frac{d(x^2 - 1)}{dx}$$

$$= (\ln 3)3^{(x^2 - 1)}(2x)$$

PRACTICE PROBLEMS:

Find $\frac{dy}{dx}$ of the following:

1. $y = \ln(2 - e^x)$

2. $y = e^{\sin(5x)}$

3. $y = 4^{(-x)}$

ANSWERS:

1. $e^x/(e^x - 2)$

2. $5e^{\sin(5x)} \cos(5x)$

3. $-(\ln 4)4^{(-x)}$

SUMMARY

The following are the major topics covered in this chapter:

1. **Derivative of a constant:**

   Theorem 1. *The derivative of a constant is zero.*

$$\text{If } y = c, \text{ then } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

2. **Derivative of a variable raised to a power:**

   Theorem 2. *The derivative of the function $y = x^n$ is given by $\frac{dy}{dx} = nx^{n-1}$, if n is any real number.*

3. **Derivative of the sum of two or more functions:**

   Theorem 3. *The derivative of the sum of two or more differentiable functions of x is equal to the sum of their derivatives.*

   If two functions of x are given, such that $u = g(x)$ and $v = h(x)$, and also $y = u + v = g(x) + h(x)$, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

4. **Derivative of the product of two or more functions:**

   Theorem 4. *The derivative of the product of two differentiable functions of x is equal to the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function.*

$$\text{If } y = uv, \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

   This theorem can be extended to three or more functions.

$$\text{If } y = uvw, \text{ then } \frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}.$$

5. **Derivative of the quotient of two functions:**

Theorem 5. *At a point where the denominator is not equal to zero, the derivative of the quotient of two differentiable functions of x is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

$$\text{If } y = \frac{u}{v}, \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

6. **Derivative of a function raised to a power:**

Theorem 6. *The derivative of any differentiable function of x raised to the power n, where n is any real number, is equal to n times the polynomial function of x to the (n − 1) power times the derivative of the polynomial itself.*

If $y = u^n$, where u is any differentiable function of x, then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

7. **Derivative of a function in radical form:** To differentiate a function containing a radical, replace the radical by a fractional exponent; then find the derivative by applying the appropriate theorems.

8. **Derivative of a function using the chain rule:**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where the variable $y = f(u)$ is a differentiable function of u and $u = g(x)$ is differentiable function of x.

9. **Derivative of an inverse function:**

Theorem 7. *The derivative of an inverse function is equal to the reciprocal of the derivative of the direct function.*

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ or } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

10. **Derivative of an implicit function:** In equations containing x and y, if an equation of y is not solved for, then y is called an *implicit function* of x. The derivative of each term containing y will be followed by $\frac{dy}{dx}$.

11. **Derivative of trigonometric functions:**

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$$

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}$$

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}$$

12. **Derivative of natural logarithmic functions:**

Theorem 8. *The natural logarithm* $y = \ln x$ *has the derivative*

$$\frac{dy}{dx} = \frac{1}{x} \text{ for } x > 0$$

If u is a positive differentiable function of x, then

$$\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$$

13. **Derivative of a constant to a variable power:**

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$$

$$\frac{d(a^x)}{dx} = (\ln a)a^x$$

$$\frac{d(a^u)}{dx} = (\ln a)a^u \frac{du}{dx}$$

where x is a variable, u is a function of x, a is a constant, and e is equal to 2.71828. . . .

ADDITIONAL PRACTICE PROBLEMS

Find the derivative $\frac{dy}{dx}$ of the following:

1. $f(x) = 3/4$
2. $f(x) = 3x^{-2/3}$
3. $f(x) = 6x^{-1/3} - 8x^{1/4}$
4. $f(x) = (2x^3 + 5)\,(4x^{-2} - 3)$
5. $f(x) = (x - 2)\,(5x - 2)\,(4x + 3)$
6. $f(x) = \dfrac{x^2 + 1}{1 - x^2}$
7. $f(x) = (x^5 - 3x^{-2} + 1)^{-13}$
8. $f(x) = 6\sqrt[3]{x} + 8\sqrt[4]{x}$
9. $y = 2t^5 + 1/t^5$ and $t = x^4$
10. $x = 1 + (y + 4)^{3/4}$
11. $2x + yx - xy^3 = 4$
12. $f(x) = \sqrt{\cot (x^2)}$
13. $f(x) = 2 \sec (\sqrt{x}) + 2\sqrt{\csc x}$
14. $f(x) = (x^3 + 4)^{1/3} \tan (3x)$
15. $y = [\ln (x^3)]^4$
16. $y = e^{-1/x} + x^e$
17. $y = 7^{3(\ln x)}$

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 0

2. $-2x^{-5/3}$

3. $-2x^{-4/3} - 2x^{-3/4}$

4. $-18x^2 + 8 - 40x^{-3}$

5. $60x^2 - 66x - 20$

6. $\dfrac{4x}{(1 - x^2)^2}$

7. $-13(x^5 - 3x^{-2} + 1)^{-14}(5x^4 + 6x^{-3})$

8. $2x^{-2/3} + 2x^{-3/4}$

9. $40x^{19} - 20x^{-21}$

10. $(4/3)\ (y + 4)^{1/4}$

11. $\dfrac{y^3 - y - 2}{x - 3xy^2}$

12. $-x[\cot\ (x^2)]^{-1/2}[\csc^2(x^2)]$

13. $x^{-1/2} \sec\ (\sqrt{x}) \tan\ (\sqrt{x}) - \sqrt{\csc x}\ \cot\ x$

14. $3(x^3 + 4)^{1/3} \sec^2(3x) + \tan\ (3x)\ (x^3 + 4)^{-2/3}\ x^2$

15. $\dfrac{12[\ln\ (x^3)]^3}{x}$

16. $e^{-1/x}\ \dfrac{1}{x^2} + ex^{e-1}$

17. $(\ln\ 7)\ 7^{3(\ln\ x)}\ \dfrac{3}{x}$